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CONDUCTION IN NONGRAY RADIATING GASES

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NOMENCLATURE

A_i, total band absorptance of the *i*th band;
A_{oi}, correlation parameter for the *i*th band;
B_i^2, correlation parameter for the *i*th band;
C_{oi}^2, correlation parameter for the *i*th band;
e_ω, Planck's function;
$e_{\omega i}$, Planck's function evaluated at the center of the *i*th band;
E_n, exponential integral;
k, thermal conductivity;
P, pressure;
P_{ei}, equivalent broadening pressure for the *i*th band;
q_c, conductive heat-transfer rate;
q_r, radiative heat-transfer rate;
T, temperature;
u_i, u_i^+, u_i^-, dimensionless coordinates for the *i*th band; C_{oi}^2Py, $\frac{3}{2} C_{oi}^2P\delta(\xi + \eta)$ and $\frac{3}{2} C_{oi}^2P\delta(\xi - \eta)$ respectively;
x, dummy variable for y;
y, coordinate normal to plates.

Greek symbols
δ, distance between plates;
ε, wall emittance;
η, dummy variable for ξ;
θ, dimensionless temperature ratio, T/T_2;
κ_ω, spectral absorption coefficient;
κ_P, Planck mean coefficient;
κ_R, Rosseland mean coefficient;
ξ, dimensionless coordinate, y/δ;
ρ, wall reflectance.

Subscripts
1, lower wall;
2, upper wall.

The gray-gas approximation has been extensively used to study the interaction of gaseous radiation with the other modes of heat transfer. However, there have been very few investigations, for example [1] and [2], which have dealt with the validity of the gray-gas model. This note is concerned with the application of a wide-band nongray gas model to conduction–radiation interaction in a static horizontal gas layer bounded by two walls. Gille and Goody [3] have also studied combined conduction and radiation in an ammonia gas layer; however, their investigation was primarily concerned with the stability of horizontal gas layers. The formulation of the radiative terms used here in the nongray analysis is similar to the analyses presented in [1] and [3]. The primary purpose of this note is to present representative results from a nongray analysis and to compare these to the gray-gas model. The specific problem to be treated is carbon dioxide bounded by two plates maintained at a moderate temperature difference near room temperature. The limiting surface conditions of two black walls, and of one black wall with one perfectly reflecting diffuse gray wall will be treated (see insert in Fig. 1). The two cases differ significantly in that the

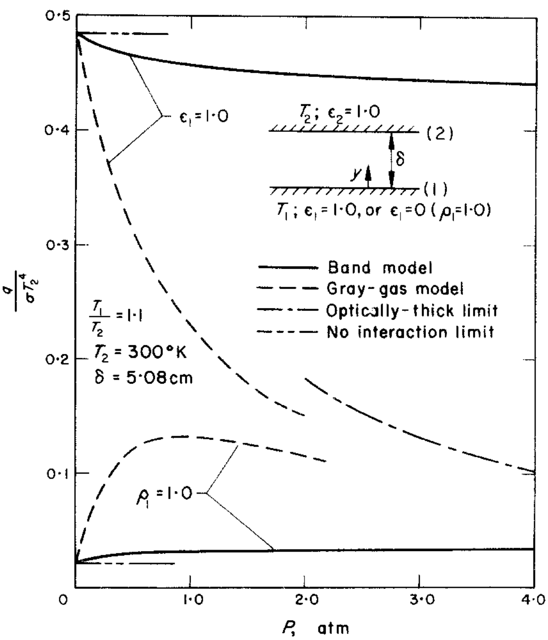

FIG. 1. Comparison of the total heat transfer for CO_2, $\delta = 5{\cdot}08$ cm.

reflecting wall problem allows no net transport of radiation between the walls due to the windows in the spectrum.

The governing energy equation is given by

$$\frac{d^2\theta}{d\xi^2} = \frac{\delta\sigma T_2^3}{k}\frac{d}{d\xi}\left[\frac{q_r}{\sigma T_2^4}\right] \tag{1}$$

with the boundary conditions

$$\xi = \frac{y}{\delta} = 0: \quad \theta = \frac{T}{T_2} = \frac{T_1}{T_2}$$
$$\xi = 1{\cdot}0: \quad \theta = 1{\cdot}0 \tag{2}$$

where q_r is the spectral radiant flux integrated over all wave numbers

$$q_r = \int_0^\infty q_{r\omega}\,d\omega. \tag{3}$$

In this analysis, it will be assumed that the gas properties are independent of temperature; i.e. the analysis is only meaningful for small temperature differences. Also, the properties will be evaluated at $T_2 = 300°K$ and only the 15 μ and 4·3 μ bands of CO_2 will be included. To solve equation (1), it is necessary to formulate an expression for q_r. This will be accomplished using two models; the gray-gas model and the exponential-band model [4].

The gray-gas model assumes an average absorption coefficient over the entire spectrum. Although the Planck mean absorption coefficient is actually an emission coefficient, it is chosen as the representative average absorption coefficient in the gray analysis. Following procedures given in [5], the mean Planck coefficient for the 15 μ and 4·3 μ fundamentals of CO_2 at 300°K is

$$\kappa_P/P = (0{\cdot}374)\ \text{cm}^{-1}\,\text{atm}^{-1}$$

where P is the pressure. Employing the exponential kernel approximation

$$E_3(t) \approx \tfrac{1}{2}e^{-\frac{3}{2}t}$$

and the expression for q_r given in [6], the following set of gray-gas equations are obtained.

$$\frac{d^2\theta}{d\xi^2} = \frac{3\kappa_P\delta^2\sigma T_2^3}{2k}\left\{(\varepsilon_1 - 1)\int_0^1 \frac{d\theta^4}{d\eta}\exp[-\tfrac{3}{2}\kappa_P\delta(\xi + \eta)]\,d\eta + \int_0^\xi \frac{d\theta^4}{d\eta}\exp[-\tfrac{3}{2}\kappa_P\delta(\xi - \eta)]\,d\eta - \int_\xi^1 \frac{d\theta^4}{d\eta}\exp[-\tfrac{3}{2}\kappa_P\delta(\eta - \xi)]\,d\eta\right\} \tag{4}$$

$$q_r/\sigma T_2^4 = -\left\{(\varepsilon_1 - 1)\int_0^1 \frac{d\theta^4}{d\eta}\exp[-\tfrac{3}{2}\kappa_P\delta(\xi + \eta)]\,d\eta + \int_0^1 \frac{d\theta^4}{d\eta}\exp[-\tfrac{3}{2}\kappa_P\delta|\xi - \eta|]\,d\eta\right\} \tag{5}$$

$$q_c/\sigma T_2^4 = \frac{-k}{\delta\sigma T_2^3}\frac{d\theta}{d\xi}. \tag{6}$$

The exponential kernel approximation is used in the gray analysis only for the purpose of consistency with the nongray analysis.

Turning to the band model, the spectral radiant flux $q_{r\omega}$ is given in [6]. Integrating this by parts, equation (3), coupled with the exponential kernel approximation and the assumption that the gradient of Planck's function de_ω/dy is constant over the band width, yields

$$q_r = \varepsilon_1 \sum_j \int_{\Delta\omega_j} (e_{\omega 1} - e_{\omega 2})\, d\omega + (1 - \varepsilon_1) \sum_i \int_0^\delta \frac{de_{\omega i}}{dx} \times \left\{ \int_{\Delta\omega_i} \exp\left[-\tfrac{3}{2}\kappa_\omega (y + x)\right] d\omega \right\} dx - \sum_i \int_0^\delta \frac{de_{\omega i}}{dx} \left\{ \int_{\Delta\omega_i} \exp\left[-\tfrac{3}{2}\kappa_\omega |y - x|\right] d\omega \right\} dx \quad (7)$$

where j denotes the windows and i the bands of the spectrum. Equation (7) can be recast in terms of the total band absorptance

$$\int_{\Delta\omega_i} (1 - \exp(-\kappa_\omega y))\, d\omega$$

which yields

$$q_r = \varepsilon_1(e_1 - e_2) - (1 - \varepsilon_1) \sum_i \int_0^\delta \frac{de_{\omega i}}{dx} A_i[3(x + y)/2]\, dx + \sum_i \int_0^\delta \frac{de_{\omega i}}{dx} A_i[3|x - y|/2]\, dx. \quad (8)$$

Differentiating equation (8) with respect to y and inserting this result into equation (1) gives

$$\frac{d^2\theta}{d\xi^2} = \frac{3}{2} \frac{\delta^2 \sigma T_2^3}{k} \sum_i \left\{ (\varepsilon_1 - 1) \int_0^1 \frac{d}{d\eta}(e_{\omega i}/\sigma T_2^4) \times A_i'[\tfrac{3}{2}\delta(\xi + \eta)]\, d\eta + \int_0^\xi \frac{d}{d\eta}(e_{\omega i}/\sigma T_2^4) A_i'[\tfrac{3}{2}\delta(\xi - \eta)]\, d\eta - \int_\xi^{1\cdot 0} \frac{d}{d\eta}(e_{\omega i}/\sigma T_2^4) A_i'[\tfrac{3}{2}\delta(\eta - \xi)]\, d\eta \right\} \quad (9)$$

where $A'(y)$ indicates the derivative of $A(y)$ with respect to y.

In order to solve equation (9), it is necessary to have a continuous representation of A and A'. Tien and Lowder [7], using the results of Edwards and Menard [4], arrived at such a correlation

$$\bar{A}_i = A_i/A_{oi} = \ln \left\{ u_i f(\beta_i) \left[\frac{u_i + 2}{u_i + 2f(\beta_i)} \right] + 1 \right\} \quad (10)$$

where $u_i = C_{oi}^2 P y$, $\beta_i = B_i^2 P_{ei}$ and $f_i(\beta_i) = 2{\cdot}94\ [1 - \exp(-2{\cdot}6\ \beta_i)]$. The parameters A_{oi}, B_i^2 and C_{oi}^2 are correlation parameters related to the spectroscopic properties of the ith band and P_{ei} is the equivalent broadening pressure. Employing the general form of equation (10) for constant properties, the energy equation is

$$\frac{d^2\theta}{d\xi^2} = \frac{3}{2} \frac{\delta \sigma T_2^3}{k} \sum_i C_{oi}^2 P \delta \left\{ (\varepsilon_1 - 1) \int_0^1 \frac{d}{d\eta} \left(\frac{e_{\omega i} A_{oi}}{\sigma T_2^4} \right) \bar{A}_i'(u_i^+)\, d\eta + \int_0^\xi \frac{d}{d\eta} \left(\frac{e_{\omega i} A_{oi}}{\sigma T_2^4} \right) \bar{A}_i'(u_i^-)\, d\eta - \int_\xi^1 \frac{d}{d\eta} \left(\frac{e_{\omega i} A_{oi}}{\sigma T_2^4} \right) \bar{A}_i'(-u_i^-)\, d\eta \right\} \quad (11)$$

where $u_i^+ = \frac{3}{2} C_{oi}^2 P \delta(\xi + \eta)$ and $u_i^- = \frac{3}{2} C_{oi}^2 P \delta(\xi - \eta)$. The radiative heat flux is given by

$$q_r/\sigma T_2^4 = \varepsilon_1 \left[\left(\frac{T_1}{T_2} \right)^4 - 1 \right] + \sum_i \left\{ (\varepsilon_1 - 1) \int_0^1 \frac{d}{d\eta} \left(\frac{e_{\omega i} A_{oi}}{\sigma T_2^4} \right) \bar{A}_i(u_i^+)\, d\eta + \int_0^1 \frac{d}{d\eta} \left(\frac{e_{\omega i} A_{oi}}{\sigma T_2^4} \right) \bar{A}_i(|u_i^-|)\, d\eta \right\} \quad (12)$$

and the conductive flux by equation (6). The CO_2 band-absorptance parameters for the 15 μ and 4·3 μ bands are given in [8]. Although there is a question as to the accuracy of the parameters for the 4·3 μ band (the integrated intensity $A_o C_o^2$ differs by approximately 40 per cent from the recommended value), the influence of the 4·3 μ band for the temperature level used here (~300°K) should be small. Planck's function $e_{\omega i}$ was evaluated at $\omega = 2350\ \text{cm}^{-1}$ for the 4·3 μ band and at $667\ \text{cm}^{-1}$ for the 15 μ band.

Numerical solutions of the governing equations for both models were carried out for a range of pressures* and plate spacings for the cases $\varepsilon_1 = 1{\cdot}0$ and $\varepsilon_1 = 0$ ($\rho_1 = 1{\cdot}0$) at $T_1/T_2 = 1{\cdot}1$. Figure 1 presents the total heat transfer as a function of gas pressure for a single plate spacing $\delta = 5{\cdot}08$ cm. This figure clearly shows the inability of the gray-gas model to predict the total heat transfer. The pressure dependency which is evident in the gray model at pressures greater than one atmosphere does not appear in the band model. The solutions for both the $\varepsilon_1 = 1{\cdot}0$ and $\rho_1 = 1{\cdot}0$ gray cases approach the optically thick solution [6] which tends to pure conduction as P increases. In the black wall case, the band model shows a very small dependency on P;

* As stated, the Planck mean absorption coefficient has been chosen as the representative average absorption coefficient for lack of a better choice. Although the Planck coefficient is generally associated with optically thin radiation, the gray results are carried beyond the optically-thin range for purposes of comparison. For the problem considered in [1], it has been shown that the Planck coefficient is incorrect even in the thin limit.

the major portion of the total heat transfer takes place in the windows of the spectrum [6]. In the $\rho_1 = 1{\cdot}0$ case, the results become almost pressure independent at $P > 1$ atm; the band absorptance becomes independent of line structure as the pressure increases and the path-length range is such that $\bar{A}$ approaches the logarithmic asymptote

$$\bar{A} = \ln u \text{ and } \bar{A}'(u) = \frac{1}{u} \propto \frac{1}{P\delta}.$$

The variation of the dimensionless conductive flux at both walls for $\delta = 5{\cdot}08$ cm is illustrated in Fig. 2 where $\theta' = d\theta/d\xi$. It should be pointed out that $\theta'(y = 0)/\theta'_0$ also represents the variation of the dimensionless total heat flux for $\rho_1 = 1{\cdot}0$. As before, the gray-gas model shows a much greater pressure dependency and overestimates the effect of radiation interaction. This overestimate, for example, in the $\rho_1 = 1{\cdot}0$ case is approximately a factor of four. With two black walls, the temperature profiles are almost antisymmetric for the temperature ratio considered here.

For sake of completeness, it is also of interest to investigate the application of the Rosseland mean absorption coefficient. Since bands will always have non-optically thick regions in the wings, the definition of this coefficient is open to question [1]. Since there is no net radiation transfer in the windows of the spectrum for the $\rho_1 = 1{\cdot}0$ case, a very simple relationship can be obtained for the total heat transfer with the assumption of constant properties. Using $q_r = -(4\sigma/3\pi_R)\, dT^4/dy$ which represents the radiant flux in the band region, the expression for the total flux reduces to the expression which is normally used for the gray optically thick limit in radiation–conduction interaction [6]. Approximate values of the Rosseland coefficient were evaluated according to procedures outlined in [5]. Since the pressure dependency of the total heat flux was of prime interest and not the magnitude, the values of κ_R used here do not include the 4·3 μ band.

Figure 3 presents the ratio of the total heat flux to that for no interaction. Shown in Fig. 3 are the $\rho_1 = 1{\cdot}0$ results using the Rosseland coefficient. Presented in this form the Rosseland calculations are independent of plate spacing [6] which is not the behavior shown by the band model. The band model has a strong plate spacing dependency for the $\rho_1 = 1{\cdot}0$ problem whereas for black plates, the dependency on plate spacing is very small. The lack of spacing dependency in the $\varepsilon_1 = 1{\cdot}0$ case would be expected since the major portion of the total flux is the radiant transfer in the window regions of the spectrum. As in the gray model, the pressure dependency of the Rosseland solution is not correct. Results obtained for the band model in the pressure range $4 \leqslant P \leqslant 10$ atm show virtually no pressure dependency for these plate spacings.

The 2·7 μ, 10·4 μ and 9·4 μ bands have been neglected in this analysis. At this temperature level $T_2 = 300$°K, the inclusion of these bands should not significantly alter the results. Before extending this analysis to higher temperatures the correlation parameters for the 4·3 μ band of CO_2 should be recalculated. As a final comment, the numerical results clearly show the inadequacy of the gray-gas model for predicting heat-transfer rates.

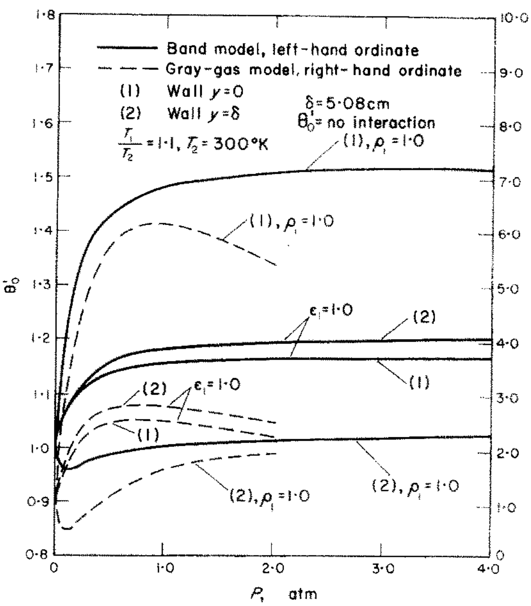

FIG. 2. Comparison of the conduction heat transfer for CO_2, $\delta = 5{\cdot}08$ cm.

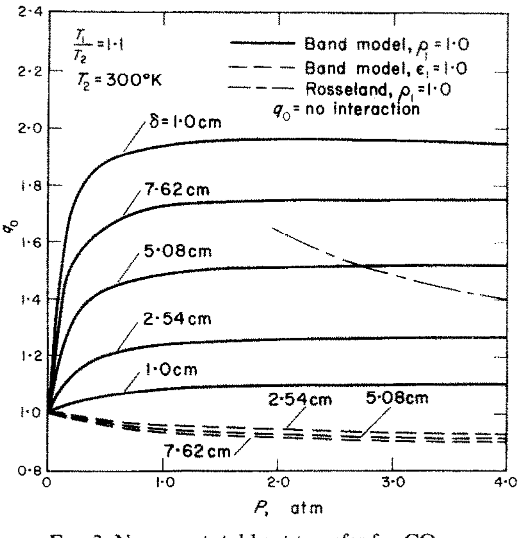

FIG. 3. Nongray total heat transfer for CO_2.

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REFLECTION OF MONODIRECTIONAL FLUX BY A COATING ON A SUBSTRATE

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NOMENCLATURE

$I(\tau, \mu)$,	intensity of radiation;
θ,	polar angle;
μ,	cosine of the polar angle θ measured from the surface normal;
τ,	optical distance;
τ_0,	optical thickness;
n_d,	refractive index of the dielectric;
ρ_D,	diffuse reflectance of the substrate;
Φ,	azimuthal angle;
Q_i,	incident flux;
q_i,	reflected attenuated flux;
$R(\mu_i', \mu_L')$,	directional reflectance due to the substrate;
$\rho_H(\mu_i')$,	directional hemispherical reflectance.

Subscripts and superscripts

c,	critical angle;
i,	incident angle;
$'$,	primes: outside the dielectric.

INTRODUCTION

Many radiative heat-transfer problems are concerned with surfaces composed of dielectric coatings on opaque metal substrates. In working with this type of surface it is desirable to predict the reflectance of the composite structure. The intent of this paper is to present an analytical approach for prediction of the reflectance of this type of coating–substrate combination.

The model employed in this analysis assumes a monodirectional flux incident on a smooth dielectric coating, which is in turn secured to a diffusely reflecting opaque substrate. The substrate is assumed to be diffuse, since a roughened or sandblasted surface would normally be used in order to insure good bonding by the coating.

TRANSPORT EQUATION

The model employed is shown in Fig. 1. The coating and substrate are assumed to be infinite in extent with a geometrical coating thickness much greater than one wavelength. The air–coating and coating–substrate interfaces are parallel. These restrictions allow the use of axial symmetry. In addition, since the problem under consideration is concerned only with reflection, the transport equation can be reduced to Beer's law, which for the axially symmetric